ON THE TRIPLE FOCUS OF A CARTESIAN.

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In his treatise on "Higher Plane Curves," Salmon says: "If I and J be each of them a cusp, then the tangent at I or J counts three times among the I or J tangents; and there are from each point $n - g - 3$ other tangents. The $(n - g)^2$ foci are then made up of one which counts as nine, of $(n - g - 3) + (n - g - 3)$ which each counts as three, and $(n - g - 3)^2$ single foci. Of these last $(n - g - 3)$ are real, and the only other real focus is the intersection of the tangents at I and J, which is commonly called a triple focus as counting for three among the real foci, though if we took into account imaginary as well as real foci, it ought to be regarded as a 9-tuple focus."

According to this theory, the foci of a Cartesian oval, the class of which is six, would be as follows: one 9-tuple focus, six triple, and nine single foci. Of these three single foci are real, and the only other real focus is the intersection of the cuspo-tangents, or, as it is commonly called, the triple focus.

This is on the condition that the line IJ does not count among the tangents to the curve. If IJ counts once among the tangents from I or J to the Cartesian, we get two real foci in finite regions beside the real triple focus, and also a real focus at infinity. Here a difficulty presents itself. It is a well known fact that a Cartesian Oval has three foci on its axis. In addition it has, in a certain sense, a fourth focus on the axis, namely, the point on the line which is infinitely distant. This we may show as follows:

From the circular points I, J at infinity four tangents may be drawn to a nodal bicircular quartic. Now we know that, if from a point on a curve, four tangents can be drawn to the curve, the anharmonic ratio of the four sets of tangents thus formed is constant; whence the tangents intersect in sixteen points, four points on each of four circles, which intersect each other orthogonally. Four of these foci, the intersections of conjugate lines, are real, the other twelve are imaginary. Now the inverse of a nodal bicircular quartic with respect to a focus is a Cartesian. Therefore, inverting with respect to one of the real foci, the circle containing these foci inverts into a straight line, three of the foci invert into the finite foci of the Cartesian, while the fourth focus, the centre of inversion, inverts into the infinite point on the line.

If the foci of a Cartesian follow the law given by Salmon, we shall have either three real single finite foci and one real triple focus, but no focus at infinity; or we shall have two real single finite foci, one real infinite, and one real triple focus. The first of these cases evidently does not hold. If now we

can show that a Cartesian has three real finite foci in addition to the triple focus, we have shown that the second case does not hold. This we may show as follows:

The equation of a Cartesian can readily be put into the form

$$s^2 - k^3(x-1),$$

where $s = 0$ is the equation of a circle. This form of the equation shows that the I, J points are cusps, and that the cuspidal tangents intersect in the centre of s, which is therefore the triple focus, or cuspo-focus of the Cartesian.

Writing ρ^2 for $x^2 + y^2$, we have for the equation of a complete Cartesian, a focus being taken for pole,

$$(\rho^2 - 2Bx + C^2)^2 - 4A^2\rho^2 = 0. \tag{1}$$

The equivalent form

$$\{(x-B)^2 + y^2 - B^2 - 2A^2 + C^2\}^2 - 4A^2(A^2 - C^2 + 2Bx) = 0$$

shows that $(x = B, y = 0)$ are the co-ordinates of the cuspo-focus.

We proceed to identify (1) with the vectorial form

$$m\rho' - \rho = K$$

that is, with

$$m(\rho^2 - 2cx + c^2)^{\frac{1}{2}} - \rho = K,$$

or

$$\left\{\rho^2 - \frac{2m^2cx}{m^2-1} + \frac{m^2c^2 - K^2}{m^2-1}\right\}^2 - \frac{4K^2\rho^2}{(m^2-1)^2}. \tag{2}$$

The equalities

$$A^2 = \frac{K^2}{(m^2-1)^2}, \quad B^2 = \frac{m^2c}{m^2-1}, \quad C'^2 = \frac{m^2c^2 - K^2}{m^2-1}$$

give

$$c^2 - (B^2 + C^2 - A^2)\frac{c}{B} + C^2 = 0, \tag{3}$$

the roots of (3) representing, of course, the two axial foci not at the origin. The roots are

$$\frac{(B^2 + C^2 - A^2) \pm (A^4 + B^4 + C^4 - 2B^2C^2 - 2C^2A^2 - 2A^2B^2)^{\frac{1}{2}}}{2B},$$

neither of which, in a true Cartesian, reduces to B.

Now returning to the tangents, the term "triple tangent" may be interpreted in two ways: 1. Three of the tangents to the curve have become coincident at the cusp; 2. The tangent at the cusp really contains only two of the

tangents to the curve, but is called triple tangent because it touches the curve in three coincident points. As has been shown above, the first interpretation does not account for all the real foci of the Cartesian. We are thus forced to the second interpretation. This, however, shows that the foci of a Cartesian do not follow Plücker's law ; and gives the following distribution of I or J tangents : Two coincident cuspidal tangents, the line IJ, and three other distinct tangents. According to this our foci are as follows : One which counts for four, eight double, and sixteen single foci. Of these foci, four single are real, three of them finite and one infinite, and the only other real focus is the quadruple one, which should be called double as counting for two among the real foci. Thus the focus which is generally considered a triple focus is really only a double focus. The single foci correspond to the sixteen foci of the bicircular quartic. As their properties and distribution are well known, nothing further need to be said about them.

Summing up these results, we see that the term " triple tangent " does not apply to the Cartesian oval, in connection with the focal properties, in the sense in which it is used in the general theory of foci, there being four tangents from each cusp in addition to the cuspidal tangents, while the class of the Cartesian is only 6. In view of this it seems to me more fitting to use the terms " cuspo-tangent " and " cuspo-focus," since these terms are not misleading as are the terms triple tangent and triple focus.